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Dissilient interpersonal influences in social network analysis

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Abstract

Influence models are currently studied by researchers working in the field of social network analysis. The basic assumption in the existing literature is that the degree of interpersonal influence among two experts can be determined in a single meeting. The second presumption is that the degree of interpersonal influence determined in this meeting is stagnant and will not change in the subsequent meetings. In this paper, we assert that degree of interpersonal influence among two or more experts can change over time. One may not find the other person equally assertive in the second or third meeting. The idea is that it may take several meetings to finally be able to declare the degree to which an expert is influenced by others. We define a composition function for the matrix of interpersonal influence and use it to define the evolution process that this matrix goes through after every meeting. We also state the conditions under which the matrix of interpersonal influence converges in the long run.

Keywords: Social network analysis; Social engagement; Influence shift; Aplomb decision maker; Dissilient interpersonal influences

1. Introduction

Social influence network (SIN) is an evolving research field. Influence models presented in literature are yet to cater for human behavior. It is understood in these models that decision makers influence one another when they interact in a group setting [3,5]. Theoretically, degree to which each individual is influenced by the other is considered similar to the notion of first impression. Meaning that the existing models assume that individuals form an opinion about others in the very first meeting and that it does not change [19,21].

In this paper, we assert that opinion evolves over time and propose the theoretical model that incorporates for this phenomenon. This is quite interesting and applicable in today's times when even if we believed adamantly in certain things before, with the advent of a pandemic someone else could reason with us the necessity of changes in lifestyle and policies that may no longer seem realistic. For instance, I may be a firm believer of children attending school.

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But as I interact with other parents on several occasions, someone may rightfully pose their opinion of not sending children to school in the midst of COVID-19 with the explanation that schools will not be able to follow the standard operating procedures necessary to ensure children's safety, and this could make sense to me and I may change my initial opinion. This change in opinion after several interactions with sharing of information and arguments is the crux of this study.

Liang et al. [15] describe social influence as the change incurred by individuals after interacting with one another. SIN builds on the idea that there exists interdependence among actors and their actions [7,12,18,24]. Adjacency matrix denoted by $W = (w_{ij})_{m\times m}$, where $m \geq 2$ and $w_{ij} \in [0, 1]$, plays a vital role in the influence models. This matrix lists the degree of interpersonal influence among m experts such that w_{ij} is the influence of expert j on the expert i [4]. Each row of the matrix satisfies the normalization property $\sum_{j=1}^{m} \omega_{ij} = 1$ for all $i \in \{1, 2, ..., m\}$. From this matrix, the information about susceptibility to interpersonal influence is deduced for each expert in the form of matrix A. Literature on fuzzy social network analysis (SIN) is available in [16,2]. In the existing models, the matrix of interpersonal influence W is conditioned in such a way that the convergence of the influence model is guaranteed in the long run [4,6,18,23,26]. When decision makers interact with each other, they influence one another in a certain way. A common pre-existing assumption is that this degree of influence is determined in the first meeting and that it cannot change in the subsequent meetings. This setting follows the ideal of first impressions being the last.

In this paper, we challenge this perception and assert that in real life, it may take more than one meetings for a decision maker to decide the degree to which other experts have influenced him. This paper lays grounds for the idea that interpersonal influence is dissilient hence the degree to which another individual has influenced him may increase or decrease with time. This is close to the real life scenario where we may get impressed by another individual's personality in the first meeting but after getting to know them better, the influence may not be the same. Similarly, we may despise a person in our first meeting but get to like them after getting to know them better. We prove that if a person is not very susceptible to interpersonal influence, and is instead self-assured or aplomb, then he will eventually be able to make up his mind about the other person in due time. After establishing that degree of influence may change over time, we state and prove that the degree of influence can be determined in finite many meetings. This means that if the decision maker is self-assured, he will be able to determine the degree to which he has been influenced by other decision makers in finite many meetings. If the decision maker is not susceptible to interpersonal influence at all, and is completely self-assured, mathematically $w_{ii} = 1$, then this decision maker can determine the degree to which others influence him in the very first meeting and that degree will be 0 because of normalization property. In any other case, for as long as the decision maker is self-assured, the degree of influence will be determined in the long run. We will prove it in this study.

Since the matrix of interpersonal influence of aplomb decision makers is dissilient but convergent, we restate the influence model. The revised influence model considers W as dissilient and non-stagnant. We prove that this influence model converges in the long run. Our model studies decision making over a set of alternative $X = \{x_1, ..., x_n\}, n \geq 3$. These alternatives can be different policies or investment options or budget allocations etc. The decision makers have to choose the best possible alternative but their opinion is influenced by other decision makers. Therefore, they may not reach a final opinion until finite many meetings. It needs to be noted that n many opinions are dispatched by each decision maker which is why our revised influence model for dissilient W is based on each alternative x_i. So the SIN takes inputs on each alternative and since the model is convergent, it helps in finding the final outcome of opinions. These outcomes are then ranked to finalize the alternative that needs immediate attention.

The paper is arranged as follows: Section 2 states some definitions that are used in the sequel. Section 3 explains what a social influence network is and studies the matrix of interpersonal influence. Section 4 puts forward the idea of interpersonal influences that are dissilient. Then it defines the conditions under which this non-stagnant matrix W will converge implying that self-sufficient decision makers will propose a degree of how much other experts have influenced them but with more meetings, the degree may change. This change is studied using composition of fuzzy matrices. We also prove in this section that the dissilient matrix converges if the decision makers are self-assured. We then propose a revised social influence network that encompasses this notion. We prove convergence of this revised iterative scheme. This method works with each alternative under consideration. In the end we rank the final opinions to conclude which alternative needs urgent attention or is the most highlighted by decision makers. Section 5 concludes the paper and proposes some future directions.

2. Preliminaries

This section summarizes some definitions that are required to understand the work proposed in this paper. In group decision modeling, a panel of experts decides which alternative is the best to solve a problem. Each expert provides a preference intensity for all possible pair of alternatives in a non-empty and finite set $X = \{x_1, x_2, ...x_n\}$.

Definition 2.1. [25] A fuzzy preference relation R on X is defined by the membership function $\mu_R : X \times X \to [0, 1]$. The membership function $\mu_R(x_i, x_j) = r_{ij}$ is interpreted as follows:

i.: The alternative x_i is absolutely preferred over the alternative x_j if $r_{ij} = 1$.
ii.: The alternative x_j is absolutely preferred over the alternative x_i if $r_{ji} = 1$.
iii.: The alternative x_i is preferred over x_j if $r_{ij} \in (0.5, 1]$.
iv.: The alternative x_j is preferred over x_i if $r_{ji} \in (0.5, 1]$.
v.: There exists indifference between the alternatives x_i and x_j if $r_{ij} = 0.5$.

Definition 2.2. [14,17] Suppose that two fuzzy relations R and S are defined on sets A, B and C such that $R \subset A \times B$ and $S \subset B \times C$. Then, the fuzzy composition $S \circ R$ is expressed by relation from A to C as follows:

For $(x, y) \in A \times B$ and $(y, z) \in B \times C$, we have,

$$\mu_{S\circ R}(x, z) = \max_y[\min(\mu_R(x, y), \mu_S(y, z))]$$

Here, $\mu_R(x, y), \mu_S(y, z) \in [0, 1]$

Definition 2.3. [22] A fuzzy matrix $A = (a_{ij})$ is a matrix that has values belonging to the closed interval $[0, 1]$. A matrix is diagonally dominant if $a_{ii} \geq \sum_{j=1, j\neq i}^{n} a_{ij}$ for all i [10].

3. Social influence network theory

SIN theory develops an influence process for a panel of n decision makers who interact with each other in group settings [8,9,13]. The underlying assumption of the existing model, which is quite realistic, is that people revise their positions based on the influence from other decision makers. Mathematically, this is calculated by taking weighted averages of the influential members of the group as follows:

$$y_i^{(t+1)} = a_i(w_{i1}y_1^{(t)} + w_{i2}y_2^{(t)} + ... + w_{iN}y_N^{(t)}) + (1 - a_i)y_i^{(1)} \tag{3.1}$$

for $t = 1, 2, ..$ and for each $i = 1, .., n$.

The initial opinions are denoted as $y_1^{(1)}, y_2^{(1)}, ..., y_n^{(1)}$ and the opinions at time $(t - 1)$ are $y_1^{(t-1)}, ..., y_n^{(t-1)}$. The influence of other decision makers on expert i is written in set-notation as $\{w_{i1}, w_{i2}, ..., w_{in}\}$, where $0 \leq w_{ij} \leq 1$, and $\sum_{j=1}^{n} w_{ij} = 1$. Another important factor in modeling influence is the susceptibility to interpersonal influence, denoted by $0 \leq a_{ii} \leq 1$ where $a_{ii} = 1 - w_{ii}$ explains the relationship between the matrix of interpersonal influence and the matrix of susceptibility of individuals.

The system of equations represented by equation (3.1) is stated as follows.

$$y^{(t+1)} = AWy^{(t)} + (I - A)y^{(1)} \tag{3.2}$$

for $t = 1, 2, ...$ Here, $y^{(1)}$ and $y^{(t)}$ represent an $n \times 1$ column vector of initial opinions and opinions at time t respectively. Also, $W = [w_{ij}]$ is an $n \times n$ matrix of interpersonal influences such that $\sum_{j=1}^{n} w_{ij} = 1$, and susceptibilities to interpersonal influence are denoted by the diagonal matrix $A = diag(a_{11}, ..., a_{nn})$. Under suitable conditions, this process transforms the initial opinions into final opinions. In order to find the final opinion, we wish to see the equilibrium point; a value after which the opinion does not change significantly anymore in the long run. For this purpose, we use equation (3.2) and apply limit as t approaches infinity.

We have, $\lim_{t\to\infty} y^{(t+1)} = \lim_{t\to\infty}(AWy^{(t)} + (I - A)y^{(1)})$ which implies that

$$(I - AW)y^{(\infty)} = (I - A)y^{(1)}$$

With the assumption that $(I - AW)$ is non-singular, we have the following.

$$y^{(\infty)} = (I - AW)^{-1}(I - A)y^{(1)}.$$

Consider, $V = (I - AW)^{-1}(I - A)$, then,

$$y^{\infty} = Vy^{(1)} \tag{3.3}$$

where $V = [v_{ij}]$ is the matrix of net interpersonal effects that have transformed the initial opinions into final opinions. In the traditional setting, $0 \le v_{ij} \le 1$ and $\sum_j v_{ij} = 1$ which represents the relative weight of initial opinion of person j in determining the final opinion of person i. Consider the following example as an application of equations (3.2) and (3.3).

Example 3.1. Consider a panel of four decision makers, $E = \{e_1 = Monica, e_2 = Col.Amy, e_3 = Faisal, e_4 = Junaina\}$ who are posed the problem of whether their country should invest more in military budget with the advent of COVID-19 or not. Assume that individuals in set E express their initial opinions on this problem as follows. Here, if the degree is closer to 1, then it means that more budget should be allocated to military. So, for instance, a value of 0.6 represents that Monica believes with a strength or degree of 0.6 that more budget should be allocated to the military whereas, Junaina's opinion is closer to 0 which means that she is not in the favor of this policy.

$$y^{(1)} = \begin{pmatrix} 0.6 \\ 0.8 \\ 0.2 \\ 0.1 \end{pmatrix}$$

After they post their initial opinions, they are placed in a group setting of an interactive session. As they discuss, some individuals influence the opinions of the others. All individuals summarize the degree with which they are influenced by other decision makers in the form of an adjacency matrix of interpersonal influence which is stated as follows.

$$W = \begin{pmatrix} 0.3 & 0.2 & 0.3 & 0.2 \\ 0 & 0.7 & 0.2 & 0.1 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then according to the model stated above, opinion $y^{(2)}$ after the first meeting will be,

$$y^{(2)} = \begin{pmatrix} 0.474 \\ 0.743 \\ 0.380 \\ 0.100 \end{pmatrix}$$

Similarly, $y^{(3)}$ onwards can be found and in the long run, the final outcome can be found as $y^{(\infty)} = \begin{pmatrix} 0.46 \\ 0.73 \\ 0.35 \\ 0.1 \end{pmatrix}$.

It can be assessed in this example that Junaina was the least susceptible to interpersonal influence and hence her initial opinion did not alter even in the long run after finite many meetings. In this paper, we have referred to such a decision maker with no or little susceptibility to interpersonal influence as an aplomb decision maker.

4. Dissilient interpersonal influences

In the traditional model, as decision makers in a panel interact with one another in a group setting, it is assumed that they will influence and be influenced by each other in the very first meeting. Furthermore, the second assumption is that the matrix of interpersonal influence W formulated from this one and only interaction is static. Accordingly, in this framework, the degree of influence is stagnant and persists for the rest of the decision making process.

In this paper, we propose people influence each other to a certain degree and with more number of meetings, this degree of influence may change. Mathematically, this translates into our matrix of interpersonal influence W as no longer being static. We believe that the matrix of interpersonal influence W is not static, it in fact transforms with time and hence it is dissilient. A question that urges from this notion whether this dissilient W ever converges? We prove in our work that it does, if the decision maker is aplomb, that is, if the decision maker is not very susceptible to interpersonal influence. Equivalently, if an expert is less susceptible to interpersonal influence, then the degree with which other people influence him will converge in a few meetings. By less susceptible to interpersonal influence, we mean that the degree to which he is subject to other people's influence is less than $\frac{1}{n}$. The initial assumption of W being stagnant becomes a special case of our study.

We believe that although first impressions are integral to human interactions but they may not necessarily be the last impression [1]. More interactions among people can certainly help them decide how others have influenced them. But then there is a possibility that some people will always be wishy washy and never be able to make up their mind about others. During this research work, we noticed that theoretically speaking, people who were less susceptible to interpersonal influence could express the degree to which other had influenced them in a few meetings. In this section we prove this assertion mathematically.

Let us define the operation $\leq$ for matrices as A is less than or equal to a matrix B if and only if $a_{ij} \leq b_{ij}$ for all i, j, where A and B are finite square matrices of same dimensions. With the help of this definition, we now define convergence. Given the sequence of powers of a matrix R, if we have $R \leq R^2 \leq R^3 \leq ... \leq R^k = R^{k+1}$ for some positive integer k, then R is said to be convergent. We assert here that influence is not stagnant which means that the matrix of interpersonal influence W is subject to change with respect to time. But we are also proposing that this change is not forever. This means that with finite many meetings, a person will eventually be able to make up his mind about another individual and in the long run, this analysis will become the basis of their future relationship.

In SNA, this means that in a certain environment, an aplomb decision maker will be able to define the degree with which the other expert has influenced him. Of course there are individuals who keep switching their opinions about the other persons and can never make up their mind but we will focus our model on aplomb decision makers; who are not much susceptible to interpersonal influence. Mathematically, this means that for aplomb decision makers, $w_{ii} > w_{ij}$ for all j. This implies that the matrix of interpersonal influence W is diagonally dominant meaning that experts are not much susceptible to interpersonal influence.

Definition 4.1. A decision maker is aplomb if he is not much susceptible to interpersonal influence. Mathematically speaking, a decision maker is aplomb if $w_{ii} > \frac{1}{n}$ for all $i \in \{1, 2, ..., n\}$, where n is an integer. A matrix of aplomb decision makers will be a diagonally dominant matrix.

So far we have laid down the foundations of this model where first impressions are not necessarily the last and with every meeting, experts get to know each other better. Since these experts are aplomb, meaning that W is diagonally dominant, they will be able to make up their minds about each other in finite time. This means that the matrix W will converge in the long run and we prove this subsequently. Moreover, each meeting will bring in changes to the matrix of interpersonal influence W until it converges, these changes will be studied with the help of maxmin composition. Consider for example the following square matrix of interpersonal influence of three aplomb experts.

Example 4.2. A panel of aplomb decision makers $E = \{e_1, e_2, e_3, e_4\}$ has the following matrix of interpersonal influence. Here, $\circ$ represents max-min operation as defined in preliminaries section. Consider,

$$W = \begin{pmatrix} 0.6 & 0.2 & 0.1 & 0.1 \\ 0.3 & 0.4 & 0.2 & 0.1 \\ 0 & 0 & 0.7 & 0.3 \\ 0.3 & 0 & 0.2 & 0.5 \end{pmatrix}$$

then

$$W^2 = W \circ W$$

$$= \begin{pmatrix} 0.6 & 0.2 & 0.1 & 0.1 \\ 0.3 & 0.4 & 0.2 & 0.1 \\ 0 & 0 & 0.7 & 0.3 \\ 0.3 & 0 & 0.2 & 0.5 \end{pmatrix} \circ \begin{pmatrix} 0.6 & 0.2 & 0.1 & 0.1 \\ 0.3 & 0.4 & 0.2 & 0.1 \\ 0 & 0 & 0.7 & 0.3 \\ 0.3 & 0 & 0.2 & 0.5 \end{pmatrix}$$
$$= \begin{pmatrix} 0.6 & 0.2 & 0.2 & 0.1 \\ 0.3 & 0.4 & 0.2 & 0.2 \\ 0.3 & 0 & 0.7 & 0.3 \\ 0.3 & 0.2 & 0.2 & 0.5 \end{pmatrix}$$

Similarly,

$$W^3 = W^2 \circ W$$
$$= \begin{pmatrix} 0.6 & 0.2 & 0.2 & 0.1 \\ 0.3 & 0.4 & 0.2 & 0.2 \\ 0.3 & 0 & 0.7 & 0.3 \\ 0.3 & 0.2 & 0.2 & 0.5 \end{pmatrix} \circ \begin{pmatrix} 0.6 & 0.2 & 0.1 & 0.1 \\ 0.3 & 0.4 & 0.2 & 0.1 \\ 0 & 0 & 0.7 & 0.3 \\ 0.3 & 0 & 0.2 & 0.5 \end{pmatrix}$$
$$= \begin{pmatrix} 0.6 & 0.2 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.2 & 0.2 \\ 0.3 & 0.2 & 0.7 & 0.3 \\ 0.3 & 0.2 & 0.2 & 0.5 \end{pmatrix}$$

And

$$W^4 = W^3 \circ W$$
$$= \begin{pmatrix} 0.6 & 0.2 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.2 & 0.2 \\ 0.3 & 0.2 & 0.7 & 0.3 \\ 0.3 & 0.2 & 0.2 & 0.5 \end{pmatrix} \circ \begin{pmatrix} 0.6 & 0.2 & 0.1 & 0.1 \\ 0.3 & 0.4 & 0.2 & 0.1 \\ 0 & 0 & 0.7 & 0.3 \\ 0.3 & 0 & 0.2 & 0.5 \end{pmatrix}$$
$$= \begin{pmatrix} 0.6 & 0.2 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.2 & 0.2 \\ 0.3 & 0.2 & 0.7 & 0.3 \\ 0.3 & 0.2 & 0.2 & 0.5 \end{pmatrix}$$

Which is the same as W^3. This implies that the matrix W has converged after three meetings. The only problem that arrives from this composition is that the final matrix does not have a row sum of 1 and that is not suitable for the influence model that we have to build. We cater to this problem shortly.

Note that no such claim can be made if the matrix of interpersonal influence is not diagonally dominant or in accordance with Definition 4.1 or in short if the experts are not aplomb. (See Fig. 1.)

Now we define the influence model for the situation when interpersonal influence is dissilient. With the passage of time, decision makers get to know the other people in panel better than before and take their time to make up their mind about other decision makers provided that the decision makers are aplomb. So for $t \geq 2$, we have the following model for dissilient W and aplomb decision makers.

$$y^{(t)} = AW^{(t-1)}y^{(t-1)} + (I - A^{(t)})y^{(1)} \tag{4.1}$$

where for $T \in [2, \infty]$, we have,

$$W^{(t)} = \begin{cases} W^{(1)} & \text{if } t \leq T \\ \mathbf{W} & \text{if } t > T. \end{cases}$$

Here, $\mathbf{W} = (\mathbf{w}_{ij})$ is such that

$$\mathbf{w}_{ij} = \frac{w_{ij}^{(T)}(1 - w_{ii}^{(T)})}{\sum_{j=1}^{n} w_{ij}^{(T)}}, i \neq j$$

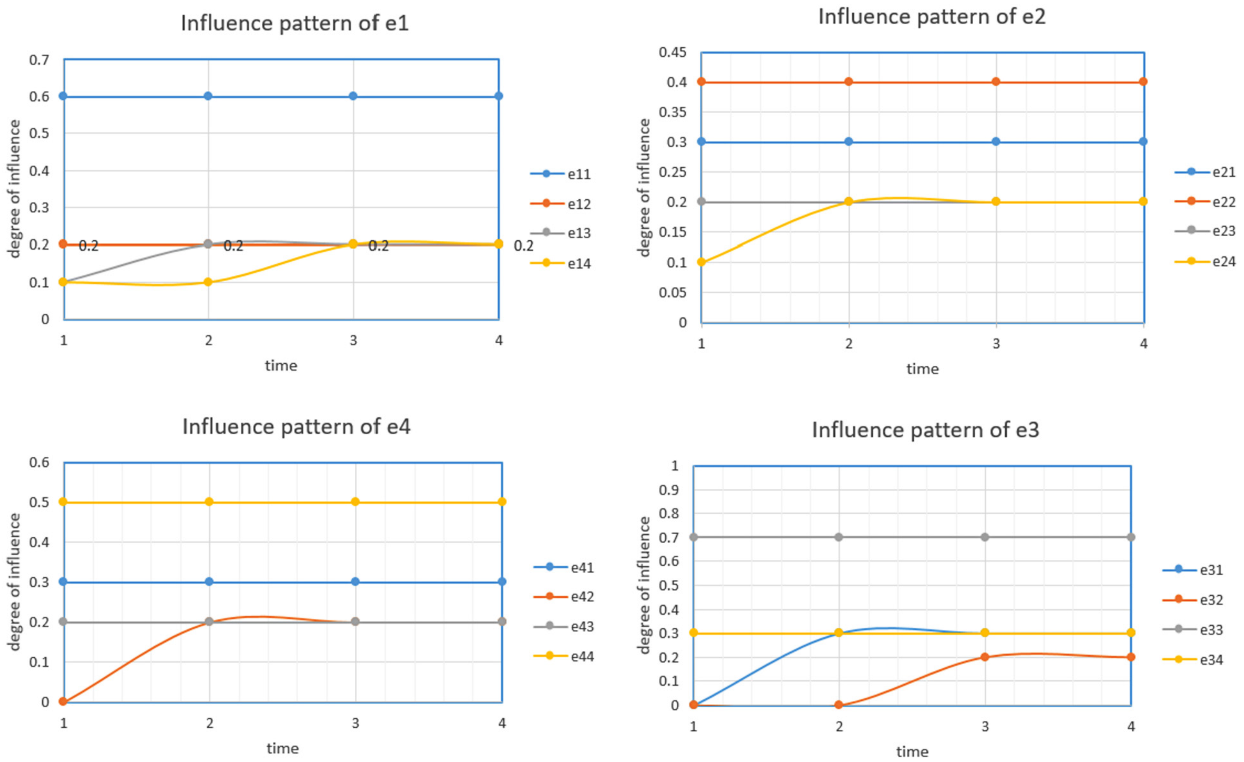

Fig. 1. Interpersonal Influence pattern of experts.

and $w_{ij}^{(T)} \in W^{(T)}$ where $W^{(T)}$ is the converged matrix. Similarly,

$$A^{(t)} = \begin{cases} A^{(1)} & \text{if } t \leq T \\ \mathbf{A} & \text{if } t > T, \end{cases}$$

where $\mathbf{A}$ is a diagonal matrix such that $\mathbf{a}_{ii} = 1 - \mathbf{w}_{ii}$ where $\mathbf{w}_{ii} = w_{ii}$. Accordingly, the converged matrix $\mathbf{W}$ for this example will be as follows.

$$\mathbf{W} = \begin{pmatrix} 0.6 & 0.133 & 0.133 & 0.133 \\ 0.25 & 0.4 & 0.171 & 0.171 \\ 0.1125 & 0.075 & 0.7 & 0.1125 \\ 0.214 & 0.142 & 0.142 & 0.5 \end{pmatrix}$$

In this section, we have asserted that the first impression may not be the last [1]. Hence, it may take some time for decision makers to make up their minds about the degree to which others have influenced them. The degree to which another decision maker has influenced you may be clearly stated after a few meetings. But there is a condition underlying this assumption which is that the decision maker is aplomb. We prove that in the long run, the matrix of interpersonal influences will converge if the panel is comprised of aplomb decision makers Definition 4.1.

Proposition 4.3. *For a matrix of aplomb decision makers, we will have $w_{ij} \leq w_{ik} \circ w_{kj}$ where w_{ij} is the $ij-th$ entry in the n by n square fuzzy matrix of interpersonal influences.*

Proof. We have,

$$w_{ij} \leq \max(\min(w_{i1}, w_{1j}), \min(w_{i2}, w_{2j}), ..., \min(w_{ii}, w_{ij}), ..., \min(w_{in}, w_{nj}))$$

That is,

$$w_{ij} \leq \max(\min(w_{i1}, w_{1j}), \min(w_{i2}, w_{2j}), ..., w_{ij}, ..., \min(w_{in}, w_{nj})) \tag{4.2}$$

This leads to two possible options. Either the right hand side of inequality (4.2) will be less than or equal to w_{ij}, in which case, $\max(\min(w_{i1}, w_{1j}), \min(w_{i2}, w_{2j}), ..., w_{ij}, ..., \min(w_{in}, w_{nj}))$ will be w_{ij}. Hence, $w_{ij} \leq w_{ij}$.

Or, some other term will be greater than w_{ij} in which case,

$$\max(\min(w_{i1}, w_{1j}), \min(w_{i2}, w_{2j}), ..., w_{ij}, ..., \min(w_{in}, w_{nj})) \geq w_{ij}$$

In either case, if the matrix is diagonally dominant we will have $w_{ij} \leq w_{ik} \circ w_{kj}$ □

Proposition 4.4. *For a matrix of aplomb decision makers W, we have that $W \leq W^2$.*

Proof. Consider

$$w^2_{ij} = \max(\min(w_{i1}, w_{1j}), ..., \min(w_{ii}, w_{ij}), ..., \min(w_{in}, w_{nj})).$$

Since W is a matrix of aplomb decision makers. Hence it is diagonally dominant, we have, $\min(w_{ii}, w_{ij}) = w_{ij}$. Therefore,

$$w^2_{ij} = \max(\min(w_{i1}, w_{1j}), ..., w_{ij}, ..., \min(w_{in}, w_{nj}))$$

This leads to two possibilities, either

$$\max(\min(w_{i1}, w_{1j}), ..., w_{ij}, ..., \min(w_{in}, w_{nj})) = w_{ij}$$

in which case $w^2_{ij} = w_{ij}$ meaning that the matrix has already converged, or,

$$\max(\min(w_{i1}, w_{1j}), ..., w_{ij}, ..., \min(w_{in}, w_{nj})) \neq w_{ij}$$

In which case,

$$\max(\min(w_{i1}, w_{1j}), ..., w_{ij}, ..., \min(w_{in}, w_{nj})) \geq w_{ij}$$

Hence, $w^2_{ij} \geq w_{ij}$.

Another way of proving this is with the help of Proposition 4.3, we have, $f^2_{ij} = \max(\min(w_{i1}, w_{1j}), ..., \min(w_{ii}, w_{ij}), ..., \min(w_{in}, w_{nj})) = w_{ik} \circ w_{kj} \geq w_{ij}$ because of Proposition 4.3. Therefore, $w_{ij} \leq w^2_{ij}$. □

Next, we prove that if decision makers are aplomb than their susceptibility to interpersonal influence will not change with time. Which means that after interactions which are mathematically represented as max-min compositions, the diagonal entries will not change. In the following, we prove that w_{ii} stays the same under the max-min operator.

Proposition 4.5. *If W is a matrix of aplomb decision makers, then their susceptibility to interpersonal influence will not change after interactions.*

Proof. For any i, we have,

$$\begin{aligned} w^2_{ii} &= w_{ii} \circ w_{ii} = \max(\min(w_{i1}, w_{1i}), .., \min(w_{ii}, w_{ii}), ..., \min(w_{in}, w_{ni})) \\ &= \max(\min(w_{i1}, w_{1i}), .., w_{ii}, ..., \min(w_{in}, w_{ni})) \end{aligned}$$

Since the matrix under consideration is diagonally dominant,

$$\max(\min(w_{i1}, w_{1i}), .., w_{ii}, ..., \min(w_{in}, w_{ni})) = w_{ii}$$

which means that w_{ii} does not change or converges to itself after each composition. □

Proposition 4.6. *If W is a diagonally dominant matrix then $W \leq W^n$ where $n > 2$.*

Proof. Note that

$$w_{ij}^{n} = \max(\min(w_{i1}^{n-1}, w_{1j}), ..., \min(w_{ii}^{n-1}, w_{ij}), ..., \min(w_{in}^{n-1}, w_{nj}))$$

Here, $\min(w_{ii}^{n-1}, w_{ij})$ will be w_{ij}. The reason is that $w_{ii}^{n-1} = w_{ii}$ since the matrix is diagonally dominant. Therefore, $\min(w_{ii}, w_{ij}) = w_{ij}$. Hence,

$$w_{ij}^{n} = \max(\min(w_{i1}^{n-1}, w_{1j}), ..., w_{ij}, ..., \min(w_{in}^{n-1}, w_{nj})) \tag{4.3}$$

The right hand side of equation (4.3) is either w_{ij} in which case $w_{ij}^{n} = w_{ij}$. Or it is not equal to w_{ij}, in which case,

$$\max(\min(w_{i1}^{n-1}, w_{1j}), ..., w_{ij}, ..., \min(w_{in}^{n-1}, w_{nj})) \geq w_{ij}$$

Thus $w_{ij}^{n} \geq w_{ij}$. □

Theorem 4.7. *If W is a diagonally dominant matrix then $W^{n-1} \leq W^{n}$.*

Proof. Can be easily derived from propositions. □

We define a compact matrix in the following definition.

Definition 4.8. [20] A fuzzy square matrix R is compact if it satisfies the condition $R \leq R^2$.

Theorem 4.9. *If W is a matrix of interpersonal influence among self-assured decision makers then the sequence of powers of the matrix $W \leq W^2 \leq ... \leq W^n \leq ..$ will converge.*

Proof. In the sequence of powers of matrix W of self-assured decision makers, we have already proved that every max-min operation results in a greater matrix. We are left to prove that sequences converge such that for some positive integer k, $R^k = R^{k+1}$. The proof is simple as according to Proposition 4.4, the self-assured matrix of interpersonal influence W is compact hence convergent due to monotonicity [11]. Moreover, it can be seen that since the matrix is compact, therefore the sequence becomes a chain. Hence, it will have a supremum. This can be done because partially ordered sets containing upper bounds for every chain necessarily as at least one maximal element. □

We proved our point that in the long run, all self-assured decision makers will state the evolved degree of interpersonal influence. Mathematically, the matrix W will converge. Hence, equation (4.1) is re-written as follows.

$$y^{(t)} = \begin{cases} A^{(1)}W^{(1)}y^{(t-1)} + (I - A^{(1)})y^{(1)} & \text{if } t \leq T \\ \mathbf{AW}y^{(t-1)} + (I - \mathbf{A})y^{(1)} & \text{if } t > T. \end{cases} \tag{4.4}$$

It is important to discuss the convergence of the iterative scheme stated in equation (4.4). In the long run as time approaches infinity,

$$\begin{aligned} \lim_{t\to\infty} y^{(t)} &= \lim_{t\to\infty} (A^{(1)}W^{(1)}y^{(t-1)} + (I - A^{(1)})y^{(1)}) \\ &= \lim_{t\to\infty} (A\mathbf{W}y^{(t-1)} + (I - \mathbf{A})y^{(1)}) \end{aligned}$$

This implies that, $y^{(\infty)} = \mathbf{AW}y^{(\infty)} + (I - \mathbf{A})y^{(1)}$. This can be stated as $(I - \mathbf{AW})y^{(\infty)} = (I - \mathbf{A})$. With the underlying condition that $(I - \mathbf{AW})$ is non-singular, we have that $y^{(\infty)} = (I - \mathbf{AW})^{-1}(I - \mathbf{A})$ which gives the final opinion.

Going back to the non-stagnant influence model, if there are m decision makers $E = \{e_1, ..., e_m\}$ who state their opinions on the set of alternatives $X = \{x_1, ..., x_n\}$. Suppose that the $k-th$ decision maker provides his opinions over X in the form of the following priority vector,

$$P_k = \begin{pmatrix} p_1^k \\ \dots \\ \dots \\ \dots \\ p_n^k \end{pmatrix}$$

Then, in order to use the model described in equation (4.4), we need to segregate information from priority vectors on the basis of each alternative. We have,

$$y_{x_i} = \begin{pmatrix} p_1^i \\ \dots \\ \dots \\ \dots \\ p_m^i \end{pmatrix}$$

to describe the preference of all the m experts over the alternative x_i. Once that is done, and we have available the $m \times m$ matrix of interpersonal influences W of self-assured decision makers, we are ready to use equation 7. According to the main contribution of the paper, the matrix W will converge with time. Also, the opinions will reach a final form. This would provide us with the final opinion of experts $y_{x_i}^{(\infty)}$ segregated according to the alternative x_i.

$$y_{x_i}^{(\infty)} = \begin{pmatrix} p_1^{i(\infty)} \\ \dots \\ \dots \\ \dots \\ p_m^{i(\infty)} \end{pmatrix}$$

Accordingly, the iterative scheme presented in 7 is re-written as follows to incorporate for finding final outcomes with respect to each alternative. Hence, for each alternative x_i, we have,

$$y_{x_i}^{(t)} = \begin{cases} A^{(1)} W^{(1)} y_{x_i}^{(t-1)} + (I - A^{(1)}) y_{x_i}^{(1)} & \text{if } t \leq T \\ \mathbf{A}\mathbf{W} y_{x_i}^{(t-1)} + (I - \mathbf{A}) y_{x_i}^{(1)} & \text{if } t > T. \end{cases} \tag{4.5}$$

Then, the overall rank given to an alternative by m experts will be calculated as $\sum_{k=1}^{m} p_k^{i(\infty)}$. Finally, the most important policy or alternative among X is calculated as $\max(\sum_{k=1}^{m} p_1^{i(\infty)}, \dots, \sum_{k=1}^{m} p_m^{i(\infty)})$ and the symbol used to declare this ranking is $\succ$. Similarly, the second best to the worst preferred alternative are found.

We use this non-stagnant influence model in the following example. There are four decision makers who have to decide which area of problems faced by the country needs immediate attention.

Example 4.10. Consider the committee of policy makers $E = \{e_1 =$ planning minister, $e_2 =$ education minister, $e_3 =$ minority minister, $e_4 =$ environmental minister$\}$ who display their opinions over the set of problems $X = \{x_1 = \mathit{education}, x_2 = \mathit{unemployment}, x_3 =$ minority rights, $x_4 =$ climate change$\}$ faced by the state of Snowland. They give their initial opinions as follows.

$$P_1 = \begin{pmatrix} 0.4 \\ 0.8 \\ 0.3 \\ 0.5 \end{pmatrix} P_2 = \begin{pmatrix} 0.6 \\ 0.4 \\ 0.7 \\ 0.2 \end{pmatrix} P_3 = \begin{pmatrix} 0.9 \\ 0.3 \\ 0.7 \\ 0.4 \end{pmatrix} P_4 = \begin{pmatrix} 0.9 \\ 0.3 \\ 0.8 \\ 0.6 \end{pmatrix}$$

We now derive information from these priority vectors pertaining to each alternative. This information can then be used as input to the non-stagnant influence model.

$$y_{x_1} = \begin{pmatrix} 0.4 \\ 0.6 \\ 0.9 \\ 0.9 \end{pmatrix} y_{x_2} = \begin{pmatrix} 0.8 \\ 0.4 \\ 0.3 \\ 0.3 \end{pmatrix} y_{x_3} = \begin{pmatrix} 0.3 \\ 0.7 \\ 0.7 \\ 0.8 \end{pmatrix} y_{x_4} = \begin{pmatrix} 0.5 \\ 0.2 \\ 0.4 \\ 0.6 \end{pmatrix}$$

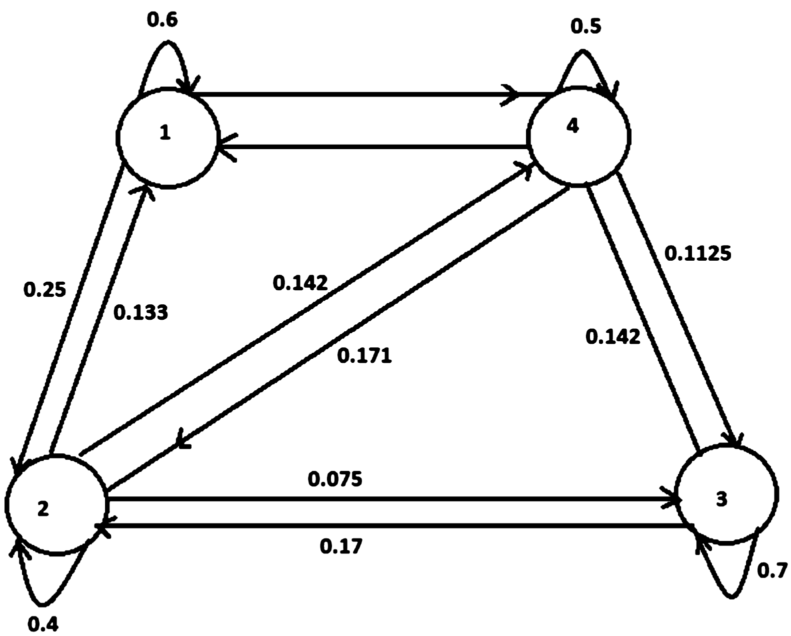

Fig. 2. Graphical representation of interpersonal influence matrix W.

Note that the matrix of interpersonal influences W is the same as the one stated in Example 4.2 and the **W** was derived above as follows.

$$\mathbf{W} = \begin{pmatrix} 0.6 & 0.133 & 0.133 & 0.133 \\ 0.25 & 0.4 & 0.171 & 0.171 \\ 0.1125 & 0.075 & 0.7 & 0.1125 \\ 0.214 & 0.142 & 0.142 & 0.5 \end{pmatrix}$$

This is represented in graph theoretic form in Fig. 2.

Accordingly,

$$y_{x_1}^{(\infty)} = \begin{pmatrix} 0.478 \\ 0.6371 \\ 0.8706 \\ 0.8109 \end{pmatrix} y_{x_2}^{(\infty)} = \begin{pmatrix} 0.7117 \\ 0.445 \\ 0.3249 \\ 0.3744 \end{pmatrix} y_{x_3}^{(\infty)} = \begin{pmatrix} 0.3788 \\ 0.632 \\ 0.685 \\ 0.712 \end{pmatrix} y_{x_4}^{(\infty)} = \begin{pmatrix} 0.484 \\ 0.329 \\ 0.4074 \\ 0.54 \end{pmatrix}$$

Finally according to the ranking method proposed in this section, alternative $x_1 \succ x_3 \succ x_2 \succ x_4$.

5. Conclusion and future work

Classical models of SNA consider the matrix of interpersonal influence as a non-stagnant. The underlying assumption is that when decision makers interact in a group setting, they can readily express the degree to which others have influenced them. The assumption goes on to include that the degree of influence cannot change with more meetings over time.

We propose in this paper that a decision maker may need more time to be able to make up their minds about others. We rule out the case where some decision makers may change their mind about others every time they meet them. We only focus on individuals who are self-assured or aplomb. We define aplomb decision makers as those who are less susceptible to interpersonal influence. We prove theoretically that if the decision makers are aplomb, they will make their mind about another decision maker in a few meetings. Which is to say that the matrix W of interpersonal influences will converge if the decision makers under consideration are dissilient.

Then, we revise the model of SNA to cater for the dissilient W which is non-stagnant. We have proved that the iterative scheme representing the social network of dissilient W is convergent as well. The problem that arises here is that the converged matrix of interpersonal influence is not normalized. A formula is stated to fix this problem as well. We have proposed examples to explain this phenomenon.

Another problem is that since we are tackling a decision making problem of more than one alternative, we have to re-write the model accordingly so that it can find the final opinion of all the experts with respect to each alternative. In the end, we rank the alternatives to conclude which alternatives need immediate attention as compared to others.

In future, we wish to also find susceptibility of each expert to interpersonal influence based on some of his physiological and psychological traits. We can rank the alternatives using the Delphi technique. Moreover, SNA can be extended to find that if artificially intelligent robots interact with each other, will they influence the decision making process of one another or not.

Declaration of competing interest

The authors declare that they have no conflict of interest.

Data availability

Data will be made available on request.

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